AD-A245 788

NPSOR-92-008

NAVAL POSTGRADUATE SCHOOL
Monterey, California

CUTTING PLANE ALGORITHMS FOR MAXIMUM PROBLEMS

Siriphong Lawphongpanich
and
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Decembei 1991

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92-03328

NAVAL POSTGRADUATE SCHOOL,
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This report was prepared for the National Science Foundation and funded by the Naval Postgraduate School, Monterey, CA.
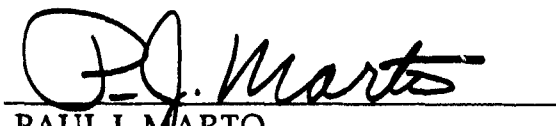
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REPORT DOCUMENTATION PAGE

Form Approved
OMB Nc. 0704-0188

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY		3. DISTRIBUTION /AVAILABILITY OF REPORT Approved for public release; distribution is unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE				
4. PERFORMING ORGANIZATION REPORT NUMBER(S) NPSOR-92-008		5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION Naval Postgraduate School	6b. OFFICE SYMBOL (If applicable) OR	7a. NAME OF MONITORING ORGANIZATION Naval Postgraduate School		
6c. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943		7b. ADDRESS (City, State, and ZIP Code) Monterey, CA 93943		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION National Science Foundation	8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN Direct Funding		
8c ADDRESS (City, State, and ZIP Code) 1800 Gth Street, NW, Washington, DC 20550		10 SOURCE OF FUNDING NUMBERS		

			PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT ACCESSION NO.

11 TITLE (Include Security Classification)

Cutting Plane Algorithms for Maximin Problems

12. PERSONAL AUTHOR(S)
Siriphong Lawphongpanich and Donald W. Hearn

13a. TYPE OF REPORT Technical	13b. TIME COVERED FROM_____ TO_____	14. DATE OF REPORT (Year, month day) December, 1991	15. PAGE COUNT 45

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Cutting plane algorithm; variational inequalities; Lagrangian
			dual; Bender's decomposition

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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20. DISTRIBUTION/AVAILABILITY OF ABSTRACT [x] UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSICIATION UNCLASSIFIED	
22a NAME OF RESPONSIBLE INDIVIDUAL S. Lawphongpanich	22b TELEPHONE (Include Area Code) (408) 646-2106	2c OFFICE SYMBOL OR/Lp

I

CUTTING PLANE ALGORITHMS FOR MAXIMIN PROBLEMS*

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Abstract

This paper unifies the development of the cutting plane algorithm for mathematical programs and variational inequalities by providing one common framework for establishing convergence. Strategies for generating cuts are provided for cases in which the algorithm yields easy and difficult subproblems. When the subproblem is easy to solve, a line search is added and a deep cut is selected to accelerate the algorithm. On the other hand, when the subproblem is difficult to solve, the problem is only solved approximately during the early iterations. This corresponds to generating cuts which are nontangential to the underlying objective function. Moreover, in the case of variational inequalities, it is shown further that the subproblem can be eliminated entirely from the algorithmic steps, thereby making the resulting algorithm especially advantageous.

I. INTRODUCTION

Many have studied the cutting plane approach for mathematical programs for some time. Cheney and Goldstein (1959) and Kelly (1960) first proposed it for convex programs. Dantzig-Wolfe (1960) (see also Dantzig, 1963) proposed the dual equivalent called generalized linear programming or column generation. Zangwill (1969) provided new viewpoints and supplied a convergence proof based on algorithmic maps. Magnanti et al. (1976) showed that generalized linear programming solves Lagrangian dual problems even when standard convexity assumptions do not hold. As a standard technique, the cutting plane approach often appears in texts such as Bazaraa and Shetty (1979).

For variational inequalities, references on the cutting plane approach are considerably less. Zuhovickii et al. (1969) (see also Auslender, 1976) were the first to apply it to variational inequalities. Nguyen and Dupuis (1984) proposed an acceleration idea and prove convergence using Zangwill's algorithmic maps.

It is well known (see, e.g., Auslender, 1976, Hearn et al., 1984 and Nguyen and Dupuis, 1984) that mathematical programs (MPs) and variational inequalities (VIs) are related and possess many similar properties. In fact, many algorithms for VI problems are based on MP algorithms (see, Harker and Pang, 1990). In spite of this relationship, the developments of the cutting plane algorithm appear different for these two areas. However, one can unify the developments by addressing MPs and VIs in a common framework. In this paper, we provide one such framework with which the cutting plane algorithm can be derived for both MPs and VIs. This framework is then used

1

to analyze existing schemes for accelerating the cutting plane algorithm. The analysis results in a common argument for proving convergence for all acceleration schemes regardless of the underlying problems and points out when each scheme might be effective.

For MPs, this paper focuses on problems derived from Benders decomposition and Lagrangian duality. For VIs, the paper considers those with finite dimension. In Section 2, it is shown that these problems can be stated as the following maximin problem:

$$w^* = \max_{u \in U} \min_{x \in X} \{f(x) + ug(x)\}$$

where U and X are nonempty and convex subsets of R^m and R^n, respectively, $f(x)$ is a continuous real-valued function define on X, $g(x)$ is a continuous vector-valued function mapping R^n into R^m, and w^* is the maximin value for the problem. For convenience, $ug(x)$ denotes the dot product of vectors u and $g(x)$. The basic cutting plane algorithm is stated in Section 3 with a convergence proof. Section 4 utilizes the framework given in Section 3 to analyze existing strategies for accelerating the algorithm. Finally, Section 5 concludes the paper.

2. INSTANCES OF MAXIMIN OPTIMIZATION

The first two instances of the maximin problem were first observed in Magnanti and Wong (1981) and they are stated here for completeness. The last instance can be found in a slightly different form in Hearn et al. (1984) and Nguyen and Dupuis (1984).

First, consider the following nonlinear program (NLP)

$$\min \ f(x)$$
$$\text{s.t.} \quad g(x) \leq 0$$
$$x \in X$$

where $f(x)$, $g(x)$, and X are as previously defined with the exception that X is additionally assumed to be compact. Then, the Lagrangian dual problem of the NLP can be stated as

$$\max_{u \geq 0} \min_{x \in X} \{f(x) + ug(x)\}$$

where u now represents the dual vector for the constraints defined by $g(x)$. This shows that Lagrangian duality leads to the desired maximin problem.

When $f(x) = cx$, $g(x) = b - Ax$, and $X = \{x : Dx \geq d, x \geq 0\}$. The above NLP reduces to the following linear program (LP):

$$\min \ cx$$
$$\text{s.t.} \quad Ax \geq b$$
$$Dx \geq d$$
$$x \geq 0,$$

and the Lagrangian dual of this LP is

$$\max_{u \geq 0} \min_{x \in X} \{cx + u(b - Ax)\}.$$

Letting $f(x) = cx$ and $g(x) = b-Ax$ yields the desired maximin problem. However, an equivalent method of obtaining the Lagrangian dual of the LP is by applying Benders decomposition to the dual of the LP which can be stated as

$$\max \quad ub+vd$$
$$\text{s.t.} \quad uA+vD \le c$$
$$u,v \ge 0.$$

Partitioning the decision variables (u,v) gives

$$\max_{u \ge 0} \max_{v \ge 0} \{ub + vd : vD \le c - uA\}.$$

By dualizing the inner maximization, the above maximin problem can be written as

$$\max_{u \ge 0} \min_{x \in X} \{ub + (c - uA)x\}, \text{ or}$$

$$\max_{u \ge 0} \min_{x \in X} \{cx + u(b - Ax)\}.$$

The last maximin problem is the same as the one obtained with Lagrangian duality.

Our last instance of a maximin problem is a variational inequality (VI) problem which can be stated as follows: find a vector $x^* \in S$ such that

$$F(x^*)(u - x^*) \ge 0 \quad \forall u \in S$$

where S is a nonempty, convex and compact subset of R^n and $F(x)$ is a vector function mapping R^n into R^n. When $F(x)$ is strongly monotone on S, i.e., $\exists \; \alpha > 0$ such that

$$(F(x) - F(u))(x - u) \ge \alpha \mid x - u \mid^2 \qquad \forall \; x, u \in S,$$

4

then the VI problem can be stated as [see Hearn et al. (1984) and Nguyen and Dupuis (1984)]

$$\max_{u \in S} \ \min_{x \in S} \ \{F(x)(x - u)\}, \ \ or \ \ \max_{u \in S} \ \min_{x \in S} \ \{F(x)x - uF(x)\}.$$

Letting $f(x) = F(x)x$ and $g(x) = -F(x)$ again yields the desired maximin problem.

3. THE CUTTING PLANE ALGORITHM

In the basic cutting plane algorithm, the optimal value of the inner minimization of the maximin problem at a given point u is denoted as $L(u)$, i.e.,

[S]: $$L(u) = \min_{x \in X}\{f(x) + ug(x)\}$$

Since $L(u)$ is defined as the pointwise minimum of a set of functions linear in u, $L(u)$ must be concave. To motivate the algorithm, the maximin problem is restated as

[P]:
$$w^* = \max \quad w$$
$$\text{s.t.} \quad w \leq f(x^j) + ug(x^j) \quad \forall x^j \in X$$
$$u \in U \text{ and } w \text{ is unrestricted.}$$

Problem [P] has an infinite number of constraints of the form

$$w \leq f(x') + ug(x')$$

which are generally referred to as *cutting planes* or simply *cuts*. To avoid generating all the cuts apriori, the algorithm initially solves an approximation of problem [P] which contains only a few cuts and obtain, say (w',u'), as a solution. To further refine the approximation and hence obtain a more accurate solution, problem [S] is solved with $u = u'$ to produce a solution x' which defines a new cut

$$w \leq f(x') + ug(x').$$

6

The approximation of problem [P] is updated and resolved with the addition of this new cut. Then, the process is repeated until an optimal solution is found. Below we formally state the algorithm for the maximin problem.

The Cutting Plane (CP) Algorithm

Step 0: Find a point $x° \in X$. Set $k = 1$ and go to Step 1.

Step 1: Solve the (k^{th}) master problem

$$\max \quad w$$

$$\text{s.t.} \quad w \leq f(x^i) + ug(x^i) \quad i = 0, ..., k-1$$

$$u \in U \text{ and } w \text{ unrestricted.}$$

Let (w^k, u^k) denote an optimal solution and go to Step 2.

Step 2: Solve the (k^{th}) subproblem

$$x^k = \underset{x \in X}{\operatorname{argmin}}\left\{f(x) + u^k g(x)\right\}$$

and $L\left(u^k\right) = f\left(x^k\right) + u^k g\left(x^k\right)$.

Step 3: If $w^k = L(u^k)$, then (u^k, x^k) solves the maximin problem.

Otherwise (i.e., $w^k > L(u^k)$), replace k by $k+1$ and go to Step 1.

In Step 1, the dual of the kth master problem is the following linear program

$$\min \quad \sum_{i=0}^{k-1} \pi_i f(x^i)$$

$$\text{s.t.} \quad \sum_{i=0}^{k-1} \pi_i g(x^i) \leq 0$$

$$\sum_{i=0}^{k-1} \pi_i = 1$$

$$\pi_i \geq 0 \quad \forall i = 0, \cdots, (k-1)$$

The CP algorithm with the master problem replaced by its dual as stated above is generally known as Dantzig-Wolfe decomposition, column generation, or generalized linear programming.

In both Benders decomposition and Lagrangian dual of an NLP, the set X is either constructed or chosen to facilitate the solving of the subproblem in Step 2 or, equivalently, problem [S]. In many cases, a closed form solution or efficient algorithm exists for the subproblem. However, such is not the case for VIs. The choice for X is restricted to be S, the feasible region of the VI problem. Moreover, the objective function of the subproblem contains the term $F(x)x$ which is at least quadratic, unless $F(x)$ is a constant vector. Therefore, the subproblem, or problem [S], does not generally admit an easy solution and the CP algorithm as stated above would be inefficient, if not ineffective, for VIs. In Section 4, a technique to overcome this difficulty is discussed.

Step 3 uses the fact that $w^k \geq L(u^k)$ which is generally known to follow from Lagrangian duality. However, the inequality can also be obtained by noting that $x^j \in X$ for all j and

$$L\left(u^k\right) = \min_{x \in X}\left\{f(x) + u^k g(x)\right\} \leq \min_j\left\{f\left(x^j\right) + u^k g\left(x^j\right): 0 \leq j \leq k - 1\right\} = w^k.$$

So, regardless of how $f(x)$ and $g(x)$ are derived, the inequality holds.

The following two lemmas are essential for the convergence of the CP algorithm. The proofs of both lemmas are similar to those in Dantzig (1963) and Magnanti et al. (1976) with the exception that all references to Lagrangian duality are eliminated. However, they are stated here for completeness and further reference.

Lemma 1: At each iteration of the CP algorithm

    (i) $w^{k-1} \geq w^k$ $\forall\, k$

    (ii) $w^k \geq w^*$ $\forall\, k$

    (iii) If $L(u^k) \geq w^k$, then $w^* = L(u^k) = w^k$.

Proof: The first result follows from the fact that the kth master problem has more cuts than the $(k-1)^{\text{st}}$. The second follows from the fact that the master problem in Step 1 contains only a finite subset of cuts in problem [P].

If $L(u^k) \geq w^k$, we have

$$w^* \leq w^k$$
$$\leq L\!\left(u^k\right) = \min_{x \in X}\left\{f(x) + u^k g(x)\right\}$$
$$\leq \max_{u \in U}\min_{x \in X}\{f(x) + ug(x)\} = w^*$$

where the first inequality follows from (ii), the second is given in (iii), and the last inequality is because of the maximization over the set U. The above series of inequalities implies that (iii) holds. $\square$

Lemma 2: If X is compact and there exists a subset $K \subseteq \{1, 2, \dots\}$ such that the subsequence $\{u^k\}_{k \in K}$ is convergent, say to the limit point, u^∞, then

$$\lim_{k \in K} w^k = w^* = L\!\left(u^\infty\right).$$

Proof: Since (w^k, u^k) solves the kth master problem and the fact that constraints in the jth master problem, with $j < k$, are always contained in the problem at iteration k, we have that

$$f\!\left(x^j\right) + u^k g\!\left(x^j\right) \geq w^k \geq w^* \quad \text{for } j = 0,1,\dots,(k-1) \tag{1}$$

where the last inequality follows from Lemma 1. Let

9

$$w^\infty = \lim_{k \in K} w^k.$$

Note that w^∞ exists because $\{w^k\}_k$ is a monotonically decreasing sequence which is bounded below by w^*. Taking the limit in equation (1) for $k \in K$, we obtain

$$f(x^j) + u^\infty g(x^j) \geq w^\infty \geq w^* \quad \text{for } j = 0,1,2,3,\ldots \tag{2}$$

By assumption, $g(x)$ is continuous and X is compact. There must exist a positive real number β such that $|g(x)| \leq \beta$ for all $x \in X$. Then,

$$\left| f(x^k) + u^k g(x^k) - f(x^k) - u^\infty g(x^k) \right| = \left| (u^k - u^\infty) g(x^k) \right| \leq \beta |u^k - u^\infty|. \tag{3}$$

Consequently, for any given $\varepsilon \geq 0$, there is a $k_1 \in K$ such that for all $k \in K$ and $k \geq k_1$, the last term in (3) is bounded by ε. Therefore,

$$\left| f(x^k) + u^k g(x^k) - f(x^k) - u^\infty g(x^k) \right| \leq \varepsilon$$

or

$$\varepsilon \geq f(x^k) + u^k g(x^k) - f(x^k) - u^\infty g(x^k) \geq -\varepsilon.$$

Examining the right inequality, we obtain

$$L(u^k) = f(x^k) + u^k g(x^k) \geq f(x^k) + u^\infty g(x^k) - \varepsilon$$

or

$$L(u^k) + \varepsilon \geq f(x^k) + u^\infty g(x^k).$$

From (2) and the definition of w^*,

$$w^* + \varepsilon \geq L(u^k) + \varepsilon \geq f(x^k) + u^\infty g(x^k) \geq w^\infty \geq w^*.$$

Since ε is arbitrary, we can conclude that

10

$$w^* = \lim_{k \in K} L(u^k) = L\left[\lim_{k \in K} u^k\right] = \lim_{k \in K} w^k$$

where the middle equality follows from the continuity of $L(u)$. ☐

To obtain a solution to the maximin problem, evaluate the function $L(u)$ at u^∞ to produce x^∞, i.e.,

$$x^\infty = \operatorname*{argmin}_{x \in X}\{f(x) + u^\infty g(x)\}.$$

Then, it follows from Lemma 2 that (x^∞, u^∞) solves the maximin problem.

We now obtain convergence for the three instances of maximin problems. For Benders decomposition of an LP, the second condition in Lemma 1 will hold after a finite number of iterations because the set X is assumed to be compact and can be expressed as a convex combination of a finite number of extreme points [see, e.g., Bazaraa et al. (1990)]. For VI problems, both U and X are the same as S which is compact. Since $u^k \in S$ for all k, $\{u^k\}_k$ must contain a convergent subsequence, thereby satisfying the hypothesis of Lemma 2. So, the CP algorithm converges for VIs. The convergence for the Lagrangian dual of an NLP follows from the theorem below.

Theorem 3: If there exists an $x^\circ \in X$ such that $g(x^\circ) < 0$ then there is a converging subsequence of $\{u^k\}_k$.

Proof: See Fisher and Shapiro (1974).

This theorem also points out that the selection of the initial solution x° in Step 0 of the CP algorithm is critical for the Lagrangian dual case, for it determines the convergence of the algorithm. Magnanti et al. (1976) provides a procedure to obtain an initial solution satisfying the condition in Theorem 3 if

one is not readily available. This procedure is akin to phase one of the two phase method in LP.

4. STRATEGIES FOR GENERATING CUTS

The two main steps in the CP algorithm consist of solving the master problem and generating a new cut. In this section, we examine existing schemes for generating new cuts.

Solving the subproblem in Step 2 of the CP algorithm is one method of generating new cuts. However, it does not distinguish between easy and difficult subproblems. When the subproblem is easy to solve, it would be advantageous to solve more subproblems in a effort to obtain better cuts, i.e., those which may lead to a reduction in the number of master problems to be solved. On the other hand, when the subproblem is difficult to solve, it may be better to cheaply obtain a legitimate cut, perhaps not necessarily tangential to $L(u)$. The two subsections below discuss these two schemes in detail.

4.1: Easy Subproblems

Hearn and Lawphongpanich (1989a) viewed the difference of two successive iterates of the CP algorithm as a direction, i.e., $d = u^k - u^{k-1}$. They showed that, when $L(u)$ is differentiable at u^{k-1}, d is an ascent direction. Otherwise, the nondifferentiability at u^{k-1} implies that the subproblem at the $(k-1)^{st}$ iteration has multiple solutions and the ascent property would depend on the choice of x^{k-1}. In any case, this observation suggests the inclusion of a line search step. Below is one version of the CP algorithm with line search.

The Cutting Plane Algorithm with Line Search (CPLS)

Step 0: Find a point $x^\circ \in X$. Let $v^\circ = 0$ and $k = 1$. Go to Step 1.

Step 1: same as before.

Step 2: same as before.

Step 3: If $w^k = L(u^k)$, then (x^k, u^k) solves the maximin problem. Otherwise (i.e., $w^k > L(u^k)$), set $d^k = u^k - v^{k-1}$.

    (i)   If $k = 1$, set $v^k = u^k$ and $k = k+1$. Go to Step 1.

    (ii) Otherwise $(k > 1)$, let

$$t_{max} = \arg\max \{L(v^{k-1} + td^k) : 0 \leq t \leq t_{up}\}$$

where t_{up} is the maximum value of t for which $v^{k-1} + td^k$ remains feasible to U. If $t_{max} \leq 1$, then pick any *nonzero* $t^k \in [t_{max}, 1]$. Otherwise, let $t^k \in [1, t_{max}]$. Set $v^k = v^{k-1} + t^k d^k$ and solve the (sub)problem

$$y^k = \arg\min_{x \in X}\{f(x) + v^k g(v)\}.$$

If y^k is not unique, select $y^k \in X(v^k)$ so that

$$f(y^k) + u^k g(y^k) \leq f(y^k) + v^k g(y^k) = L(v^k) \tag{4}$$

Set $x^k = y^k$ and $k = k+1$. Go to Step 1.

Note that CPLS produces two sets of dual iterates, u^k and v^k, where u^k denotes the solution of the master problem and v^k is a point along the direction $d^k = u^k - v^{k-1}$. Moreover, the cuts are generated at v^k instead of u^k.

In Step 3(ii), the choice of step length, t^k, in the direction d^k is inexact to allow for inaccuracy in the line search and for heuristic selection of a new cut.

14

However, setting $t^k = t_{max}$ places v^k at the maximum point and $t^k = 1$ places v^k at u^k which reduces CPLS to the basic CP algorithm.

For any allowable choice of t^k, the resulting v^k is feasible to the master problem and we have the following relationship

$$L(u^k) \le L(v^k) \le \min\{ f(x^i) + v^k g(x^i) : i = 0,...,(k-1)\} \le w^k$$

If $L(v^k) = w^k$, then CPLS would discover that (x^k, v^k) is optimal in the next iteration. Otherwise (i.e., $L(v^k) < w^k$), adding the cut

$$w \le f(x^k) + u g(x^k)$$

makes the point (w^k, u^k) infeasible to the next master problem thereby ensuring that the sequence w^k decreases monotonically.

To establish convergence for CPLS, it is assumed that either U is compact or $L(u)$ satisfies the following condition

$$[A]: \qquad \lim_{t \to \infty} L(u + td) = -\infty$$

for all $u \in U$ and for all of its direction of recession, $d \ne 0$ (see page 61 of Rockafellar, 1970). Among the three instances of maximin problems, U is compact for VI problems because $U = S$ and S is compact by the standing assumption in Section 2. For optimization problems, $U = \{u_i \ge 0 : i = 1, ..., m\}$ implies that every component of its direction of recession must be nonnegative. Then, the requirement (in Theorem 3) that there exists an $x^\circ \in X$ such that $g(x^\circ) < 0$ further implies that condition [A] holds. To verify this, note that for any $u \in U$ and any direction of recession $d \ne 0$

$$L(u + td) = \min_{x \in X} \{ f(x) + (u + td)g(x) \} \le f(x^\circ) + u g(x^\circ) + t d g(x^\circ).$$

Since $d \ge 0$ and $g(x^\circ) < 0$, $d g(x^\circ) < 0$ and taking the limit as t approaches ∞ gives

15

$$\lim_{t \to \infty} L(u + td) \le f(x^\circ) + ug(x^\circ) + \lim_{t \to \infty} tdg(x^\circ) = -\infty.$$

Under the assumption that either U is compact or condition [A] holds, the value for t_{max} in Step 3 of CPLS must be bounded. This in turn implies that v^k is also bounded whenever u^k is bounded because

$$|v^k| = |v^k - u^k + u^k| \le |v^k - u^k| + |u^k| \le \max\{1, t_{max}\} + |u^k|.$$

Then, the convergence of CPLS follows from the lemma below.

Lemma 4: If X is compact and there exists an index set $K \subseteq \{1, 2, ...\}$ such that the subsequence $\{u^k\}_{k \in K}$ converges to u^∞ then there exists a subset $K' \subseteq K$ such that $\{v^k\}_{k \in K'}$ converges to v^∞ and

$$\lim_{k \in K'} w^k = w^* = L(v^\infty)$$

Proof: Using the same argument as in Lemma 2, it can be shown that

$$f(x^j) + u^\infty g(x^j) \ge w^\infty \ge w^* \quad \forall j = 0, 1, 2, \cdots \tag{5}$$

and

$$f(x^k) + u^k g(x^k) \ge f(x^k) + u^\infty g(x^k) - \varepsilon.$$

Using inequality (4) in Step 3(ii), the right expression is bounded above as follows.

$$L(v^k) = f(x^k) + v^k g(x^k) \ge f(x^k) + u^k g(x^k) \ge f(x^k) + u^\infty g(x^k) - \varepsilon$$

or

$$L(v^k) + \varepsilon \ge f(x^k) + u^k g(x^k) \ge f(x^k) + u^\infty g(x^k).$$

From (5), we have

$$w^* + \varepsilon \ge L(v^k) + \varepsilon \ge f(x^k) + u^k g(x^k) \ge f(x^k) + u^\infty g(x^k) \ge w^\infty \ge w^*.$$

16

Since $\{u^k\}_{k \in K}$ converges, u^k must be bounded for all $k \in K$. It follows from the preceding discussion that v^k for all $k \in K$ must be bounded as well. Thus, there must exist a subset $K' \subseteq K$ such that the subsequence $\{v^k\}_{k \in K'}$ converges to v^∞. So, taking the limit of the above inequalities with respect to $k \in K'$, we have

$$w^* + \varepsilon \geq \lim_{k \in K'} L(v^k) + \varepsilon \geq L\left[\lim_{k \in K'} v^k\right] + \varepsilon \geq w^*.$$

Since ε is arbitrary, $w^* = L(v^\infty)$. $\qquad\square$

Given Lemma 4, obtaining the x component for the maximin solution corresponding to v^∞ and establishing the convergence of the three instances are the same as in the basic CP algorithm.

At the end of Step 3(ii), x^k originally defined in Step 2 is replaced with y^k. So, y^k now defines a new cut for the $(k+1)^{st}$ master problem. Moreover, y^k is not arbitrarily selected when alternate optima exist. In fact, Step 3 requires that y^k satisfies inequality (4) and the theorem below ensures that such y^k exists.

Theorem 5: In Step 3 of CPLS, there exists a y^k which solves

$$L(v^k) = \min_{x \in X}\left[f(x) + v^k g(x)\right]$$

and satisfies

$$f(y^k) + u^k g(y^k) \leq f(y^k) + v^k g(y^k) = L(v^k)$$

where u^k is part of the solution, (w^k, u^k), to the kth master problem.

Proof: Assume that $t_{max} > 1$. Denote $v^{max} = v^{k-1} + t_{max}d^k$. Then, $v^k = \beta v^{max} + (1-\beta) u^k$ for some $\beta \in (0,1)$ and by concavity of $L(u)$

$$L(v^k) \geq \beta L(v^{max}) + (1-\beta) L(u^k) \geq L(u^k).$$

Let $X(v^k)$ denote the (compact) set of solutions to the subproblem. To obtain a contradiction, assume that

$$f(x) + u^k g(x) > f(x) + v^k g(x) \qquad \text{for all } x \in X(v^k)$$

Then

$$(u^k - v^k) g(x) > 0 \qquad \text{for all } x \in X(v^k)$$

which implies that

$$\min \{ (u^k - v^k) g(x) : x \in X(v^k) \} > 0$$

Thus, the directional derivative of $L(v)$ at v^k in the direction $(u^k - v^k)$ is positive, i.e., $(u^k - v^k)$ is an ascent direction. However, v^k is a point on the line connecting v^{max} to u^k and, by the concavity of $L(u)$, moving toward u^k must decrease its value. This contradicts the statement that $(u^k - v^k)$ is an ascent direction. (The case for $t_{max} \leq 1$ is proved similarly.) $\square$

Figure 1 illustrates cuts which satisfy or are 'allowed' by inequality (4). In this case, there are an infinite number of allowable cuts and any of them would make CPLS converges.

As in Magnanti and Wong (1981), we say that the cut, $w \leq f(x')+ug(x')$, dominates or is stronger than the cut, $w \leq f(x'') + ug(x'')$ if

$$f(x') + ug(x') \leq f(x'') + ug(x'') \quad \forall u \in U$$

with a strict inequality for at least one u. We call a cut **pareto optimal** if no cut dominates it. Since a cut is determined by $x \in X$, we also say that x' dominates x'' (or x' is stronger than x'') if the associated cut is stronger, and we say that x is pareto optimal if the corresponding cut is pareto optimal.

It is interesting to note that every allowable cut in Figure 1 is pareto optimal. However, this is not always true. In general, there are allowable cuts which are dominated. Theorem 6 shows how to modify Step 3(ii) of CPLS to generate a pareto optimal cut.
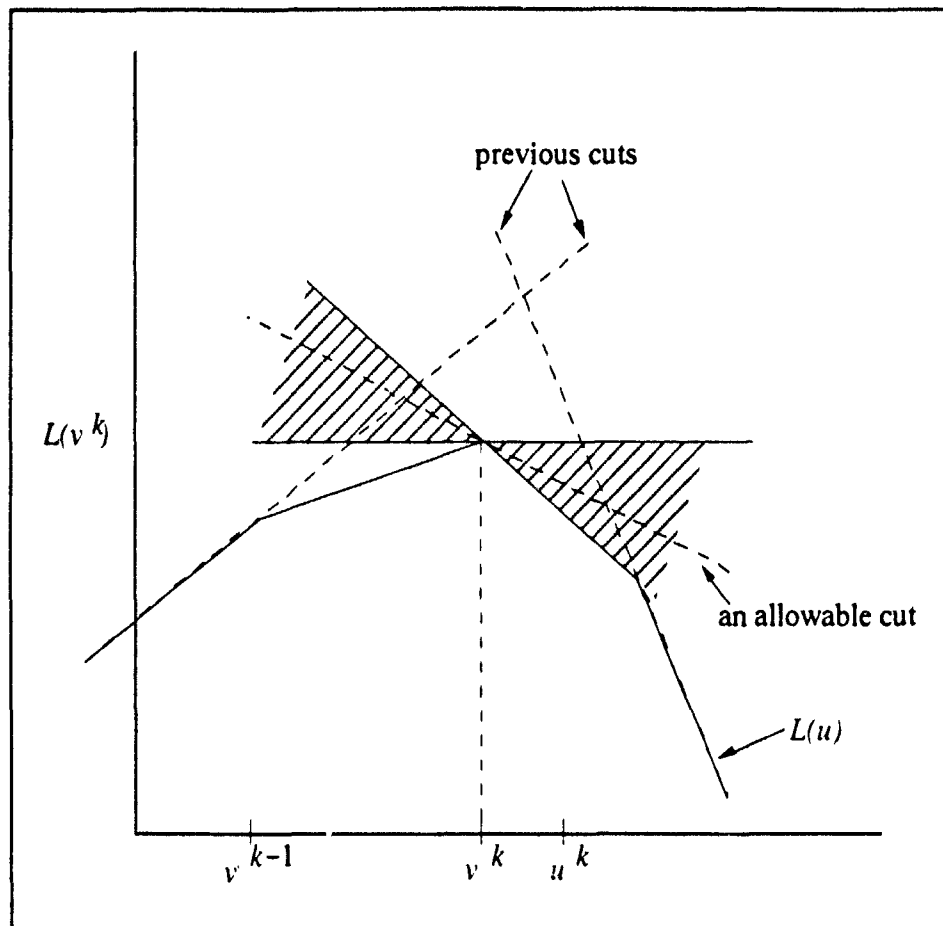

Figure 1. Allowable Cuts for Step 3 of CPLS

Theorem 6: Let $X(v^k)$ denote the solution set of the following (sub)problem.

$$\min_{x \in X}\left\{f(x) + v^k g(x)\right\}.$$

If y^k uniquely solves

$$\min_{x \in X(v^k)}\left\{f(x) + u^k g(x)\right\},$$

then y^k is pareto optimal.

Proof: Assume the contrary that y^k is not pareto optimal, i.e., there exists an $x \in X$ and $x \neq y^k$ such that

$$f(x) + ug(x) \le f\left(y^k\right) + ug\left(y^k\right) \quad \forall u \in U. \tag{6}$$

In particular, setting $u = v^k$ in (6) we obtain

$$f(x) + v^k g(x) \le f\left(y^k\right) + v^k g\left(y^k\right). \tag{7}$$

Since $y^k \in X(v^k)$,

$$f\left(y^k\right) + v^k g\left(y^k\right) \le f(x) + v^k g(x). \tag{8}$$

Combining (7) and (8) produces the following equality

$$f\left(y^k\right) + v^k g\left(y^k\right) = f(x) + v^k g(x)$$

that is, $x \in X(v^k)$. However, setting $u = u^k$ in (6) gives

$$f(x) + u^k g(x) \le f\left(y^k\right) + u^k g\left(y^k\right).$$

Since $x \neq y^k$ and y^k uniquely solves the 2nd minimization problem, this last inequality is a contradiction. $\square$

If u^k in Theorem 6 is in the relative interior of U, the uniqueness assumption for y^k can be dropped and the pareto optimality follows from Theorem 1 in Magnanti and Wong (1981). The following theorem shows that y^k in Theorem 6 defines an allowable cut.

Theorem 7: Let y^k be as defined in Theorem 6. Then

$$f(y^k) + u^k g(y^k) \leq f(y^k) + v^k g(y^k)$$

Proof: Assume the contrary that

$$f(y^k) + u^k g(y^k) > f(y^k) + v^k g(y^k).$$

Using the definition of y^k, we have that

$$\min_{y \in X(v^k)} \left\{ f(y) + u^k g(y) \right\} > f\left(y^k\right) + v^k g\left(y^k\right)$$

$$\min_{y \in X(v^k)} \left\{ f(y) + u^k g(y) \right\} - f\left(y^k\right) - v^k g\left(y^k\right) > 0.$$

Since $f(y^k) + v^k g(y^k) = L(v^k) = f(y) + v^k g(y)$ for all $y \in X(v^k)$,

$$\min_{y \in X(v^k)} \left\{ f(y) + u^k g(y) - f(y) - v^k g(y) \right\} > 0.$$

$$\min_{y \in X(v^k)} \left\{ \left(u^k - v^k\right) g(y) \right\} > 0.$$

The expression on the left side is the directional derivative of $L(v^k)$ in the direction $(u^k - v^k)$, and the inequality implies that $u^k - v^k$ is an ascent direction which is impossible for our choice of step size t^k. To see this, assume that $t^k \in [t_{max}, 1]$. This implies that v^k is on line joining v^{max} and u^k, where $v^{max} = v^{k-1} +$

21

$t_{max} (u^k - v^k)$ maximizes $L(u)$ in the direction $(u^k - v^k)$. So, moving away from v^k toward u^k can only decrease the function value, i.e., $u^k - v^k$ cannot be an ascent direction at v^k. The argument is the same for $t^k \in [1, t_{max}]$. $\square$

Theorems 6 and 7 demonstrate that it is possible to select an allowable cut which is also pareto optimal. Magnanti and Wong (1981) showed that pareto optimal cuts can accelerate the convergence of the cutting plane algorithm, particularly for Benders decomposition. Selecting an allowable cut or a pareto optimal y^k would require considerable effort in general. Hearn and Lawphongpanich (1989a & b) described a heuristic method for selecting an allowable cut. The method consists of the following two rules:

i) If $t_{max} > 1$, set $t^k = t_{max} - \varepsilon$, and

ii) If $t_{max} \leq 1$, set $t^k = t_{max} + \varepsilon$

where ε is a small positive number. Figure 2 illustrates how these rules select a cut. The function $L(u)$ is nondifferentiable at v_{max} and $t_{max} < 1$ in Figures 2(a) and (b). So, letting $t^k = t_{max} + \varepsilon$ would set v^k to the right of v_{max} by an ε amount. However, at v^k the function $L(u)$ is differentiable and the the cut generated here is the line tangential to $L(u)$. In Figure 2(a), this ε-purturbation rule chooses the only nondominated cut. In 2(b), there are an infinite number of nondominated cuts and all of which are convex combinations of the two 'extreme' cuts. In this case, the rule selects one of the extreme cuts.

previous cuts

$L(u)$

allowable cuts

w*

the pareto optimal cut

$L(u)$ feasible
region

v^k

v_{max}

u^k

(a)

previous
cuts

$L(u)$

two extreme
cuts

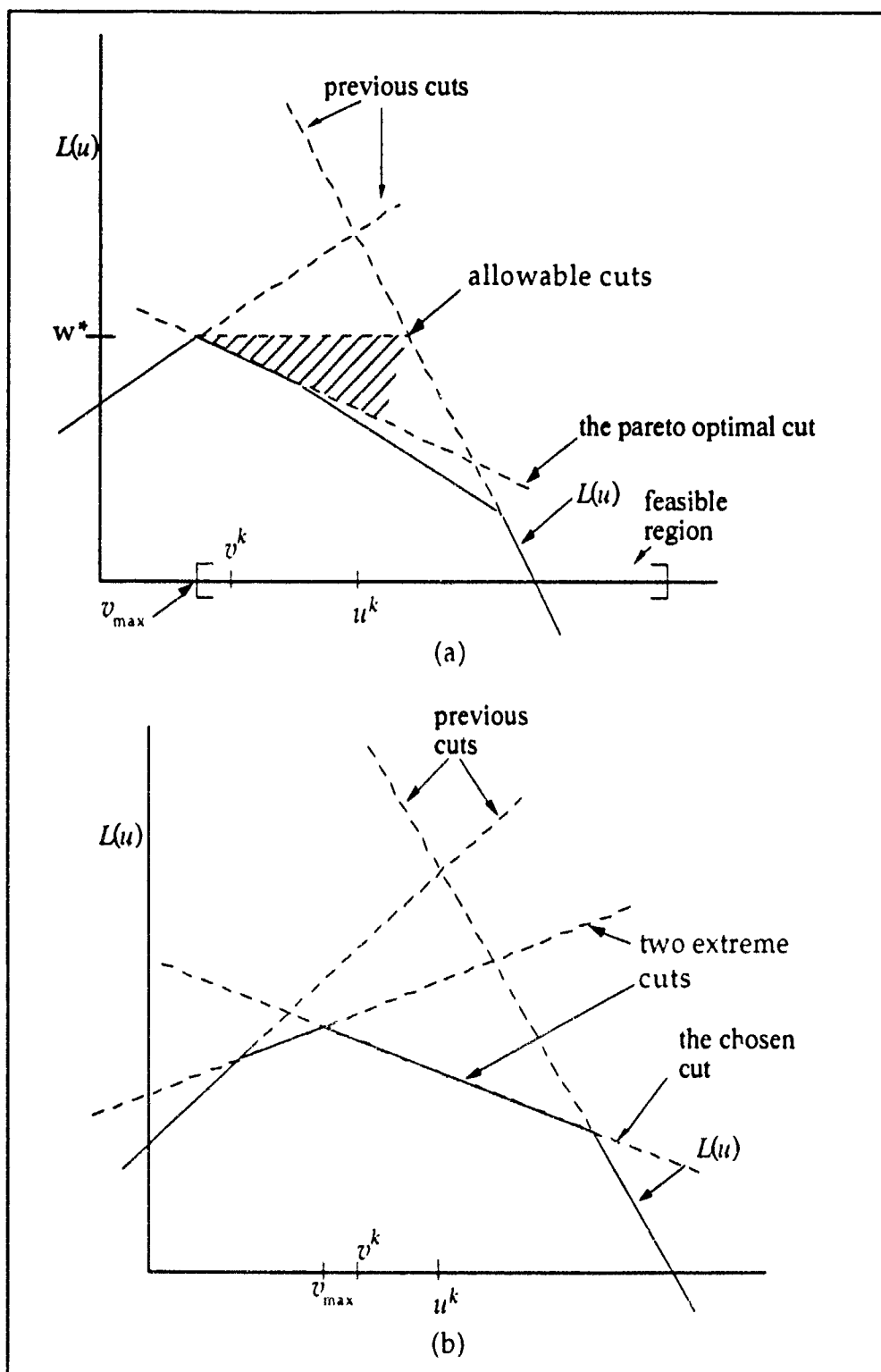the chosen
cut

$L(u)$

v^k

v_{max}

u^k

(b)

Figure 2. A Heuristic Section of Allowable Cut

4.2 Difficult Subproblems

In this situation, one alternative is to solve the subproblem only approximately while ensuring that effective cuts are still being generated. In Lagrangian relaxation or subgradient optimization (see, e.g., Fisher, 1981 and Polyak, 1969), the optimal objective function value is used to define the step length calculation which guarantees convergence. For the CP algorithm, the maximin value, w^*, analogously defines how close the approximation need to be in order to generate an effective cut. In general, the value w^* is unknown at the start of the algorithm and must be estimated to make the resulting algorithm effective. However, $w^* = 0$ for VI problems (see, e.g., Zuhovickii et al. 1969 and Auslender 1976). This allows an approximate subproblem solution to be easily obtained.

Below, we state a version of the basic CP algorithm which solve the subproblem approximately. The name 'nontangential' is due to the fact that, by approximately solving the subproblem, the generated cuts are not necessarily tangential to $L(u)$.

The Nontangential Cutting Plane Algorithm (NTCP)

Step 0: same as before.

Step 1: same as before.

Step 2: If $w^k = w^*$, then there exists an optimal solution (w^k, u^k) such that $L(u^k) = w^*$. Otherwise (i.e., $w^* < w^k$), select $x^k \in X$ such that

$$f(x^k) + u^k g(x^k) \leq w^*. \tag{9}$$

Set $k = k+1$ and go to Step 1.

24

Note that x^k in Step 2 need not be an optimal solution to the subproblem. In fact, the (subproblem) objective function value at x^k only need to be sufficiently small, i.e., no larger than w^*. This insures that (w^k, u^k) is infeasible to the $(k+1)^{st}$ master problem because

$$f(x^k) + u^k g(x^k) \le w^* < w^k.$$

From this inequality, it can be concluded that w^k is still a monotonically decreasing sequence. However, because x^k does not necessarily solve the subproblem, the cut: $f(x^k) + u^k g(x^k)$ is not necessarily tangent to $L(u)$ at u^k (see Figure 3). Nevertheless, the following results show that NTCP can still converges to the desired solution.
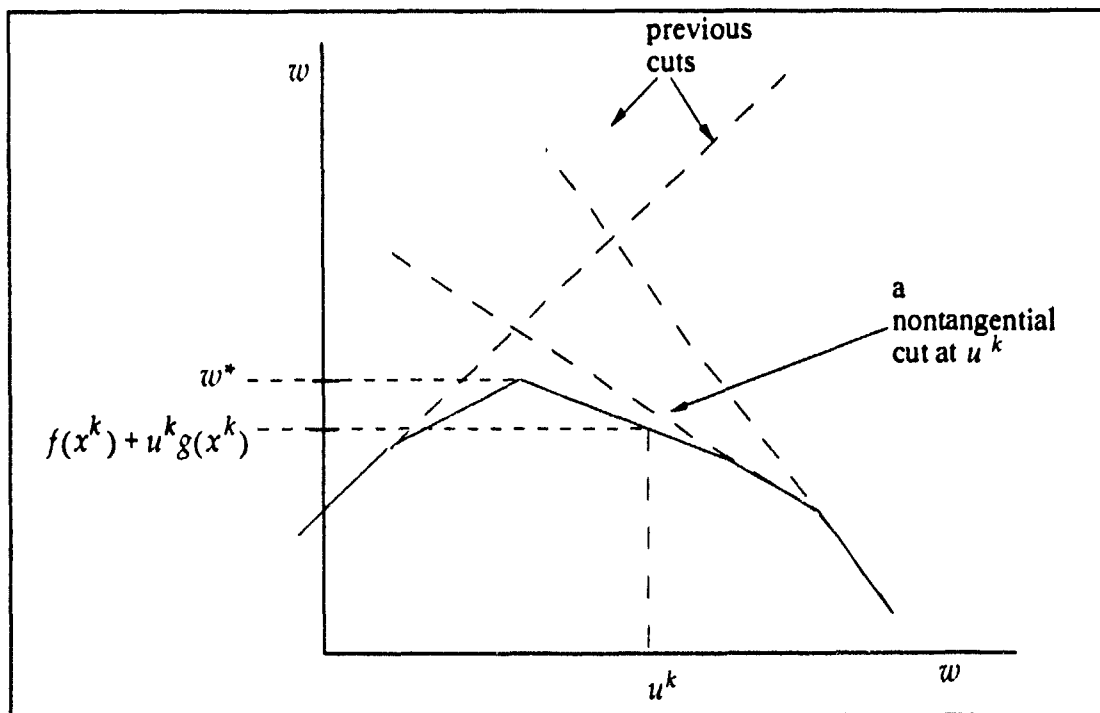

Figure 3. An Example of a Nontangential Cut

Theorem 8: If $w^k = w^*$, then there exists an optimal solution the the master

problem, (w^k, u^k), such that $L(u^k) = w^*$. That is, (x^k, u^k) solves the

maximin problem, where
$$x^k = \min_{x \in X}\{f(x) + u^k g(x)\}.$$

Proof: Assume that the theorem is not true. Then, for every pair (w^k, u^k) which

is optimal to the kth master problem, the following must hold

$$L(u^k) < \min\{f(x^i) + u^k g(x^i) : i = 0, ..., (k-1)\} = w^k = w^*. \qquad (10)$$

However, for any pair (w, u) feasible, but nonoptimal, to the kth master prob-

lem, the following also holds

$$L(u) \le \min\{f(x^i) + u g(x^i) : i = 0, ..., (k-1)\} \le w < w^k = w^*. \qquad (11)$$

Combining (10) and (11), we get
$$L(u) < w^* \quad \text{for all } u \in U$$

or
$$\max_{u \in U} L(u) < w^*.$$

This is a contradiction since w^* is the maximin value. $\square$

Lemma 9: Assume that NTCP generates an infinite sequence $\{u^k\}$. If X is com-

pact and there exists a subset $K \subseteq \{1, 2, \dots \}$ such that the subse-

quence $\{u^k\}_{k \in K}$ is convergent, say to the limit point, u^∞, then

$$w^* = \lim_{k \in K} w^k$$

Proof: Using the same argument as in Lemma 2, it can be shown that for k

sufficiently large and $k \in K$

26

$$w^* \leq w^\infty \leq f(x^k) + u^\infty g(x^k) \leq f(x^k) + u^k g(x^k) + \varepsilon. \qquad (12)$$

However, by the choice of x^k in Step 2 of NTCP

$$f(x^k) + u^k g(x^k) \leq w^* \qquad (13)$$

Combining (12) and (13) yields the following

$$w^* \leq w^\infty \leq f(x^k) + u^\infty g(x^k) \leq f(x^k) + u^k g(x^k) + \varepsilon \leq w^* + \varepsilon. \qquad (14)$$

Since ε is arbitrary, we can conclude that

$$w^* = \lim_{k \in K} w^k$$

□

To address the convergence of $\{u^k\}_K$ to u^*, where

$$w^* = \min_{x \in X} \{f(x) + u^* g(x)\},$$

define the following

i) $\quad X^\infty \quad = \quad \{x^0, x^1, x^2, \ldots\}$

ii) $\quad [X^\infty] \quad = \quad$ the closure of X^∞

iii) $\quad L^k(u) \quad = \quad \min_{i \in \{0,1,\ldots,(k-1)\}} \{f(x^i) + u g(x^i)\}$

iv) $\quad L^\infty(u) \quad = \quad \min_{x \in [X^\infty]} \{f(x) + u g(x)\}$

From (iii) and (iv), it is clear that

$$L^\infty(u) \leq L^{k+1}(u) \leq L^k(u) \qquad \forall u \in U,$$

and it can be shown that

$$\lim_{k \to \infty} L^k(u) = L^\infty(u) \quad \forall u \in U.$$

In other words, $\{L(u^k)\}$ is a monotonic sequence of functions which converges pointwise to $L^\infty(u)$. Furthermore, note that

$$u^k = \arg\max_{u \in U} L^k(u).$$

If the subsequence $\{u^k\}_K$ converges to u^∞, then it follows from Theorem 3.7 of Wets (1983) [see also the results in Wets, 1980] that

$$u^\infty = \arg\max_{u \in U} L^\infty(u),$$

and (14) also implies that

$$w^* = L^\infty(u^\infty) = \lim_{k \in K} w^k.$$

Lemma 10: Under the same assumptions as in Lemma 9, u^* solves the following problem:
$$\max_{u \in U} L^\infty(u).$$

Proof: Note that

$$w^* = \min_{x \in X} \{f(x) + u^* g(x)\} \quad \leq \quad \min_{x \in [X^\infty]} \{f(x) + u^* g(x)\}$$

$$\leq \quad \min_{i \in \{0,1,\ldots,(k-1)\}} \{f(x^i) + u^* g(x^i)\}$$

$$\leq w^k$$

where the first inequality follows from the fact that $[X^\infty]$ is a subset of X, the second from the fact that $\{0,1,2,\ldots,(k-1)\}$ is a subset of $[X^\infty]$, and the last from the fact that u^* does not necessarily solve the kth master problem.

The above sequence of inequalities can be summarized as follows

$$w^* \leq L^\infty(u^*) \leq w^k.$$

By taking the limit as $k \to \infty$, $k \in K$, and invoking Lemma 9, we have that

28

$$L^{\infty}(u^*) = w^* = \max_{u \in U} L^{\infty}(u)$$

and the proof is complete. ☐

An immediate consequence of Lemma 10 is that if the problem

$$\max_{u \in U} L^{\infty}(u)$$

admits a unique solution, then $\{u^k\}_K$ must converge to u^*. Given Lemma 9 and 10, the convergence of NTCP for the three instances of maximin problem can be established as in the case of the basic CP algorithm except for Benders decomposition. To insure that the NTCP algorithm for Benders decomposition converges *finitely*, the x^k chosen in Step 2 of NTCP must also be an extreme point of the region X.

It is interesting to specialize NTCP to VI problems. Because of the structure of VIs, the task of selecting an x^k in Step 2 is much easier and the stopping rule and the convergence result can both be strengthened. Recall that the maximin problem for VIs takes the following form

$$w^* = \max_{u \in S} \min_{x \in S} \{F(x)x \cdot F(x)u\}$$

where S is nonempty, convex and compact subset of R^n and $F(x)$ is a vector valued function mapping R^n into R^n. Moreover, $F(x)$ is further assumed to be strongly monotone which implies that there is a unique solution, u^*, to the VI problem, i.e.,

$$F(u^*)(x \cdot u^*) \geq 0 \qquad \forall x \in S.$$

Below, we state the version of NTCP algorithm for VIs.

The Cutting Plane Algorithm for Variational Inequalities (CPVI)

Step 0: Let $x^\circ \in S$. Set $k = 1$ and go to Step 1.

Step 1: Solve the (k^{th}) master problem

$$\begin{array}{lll} \max & w & \\ \text{s.t.} & w \leq F(x^i)x - F(x^i)u & i = 0, ..., (k-1) \\ & u \in S & \end{array}$$

Let (w^k, u^k) be an optimal solution and go to Step 2.

Step 2: If $w^k = 0$, stop and x^i for some $i \in \{0, ..., (k-1)\}$ is a solution. Otherwise, select $x^k \in S$ such that

$$F(x^k)(x^k - u^k) \leq 0.$$

Set $k = k+1$ and go to Step 1.

The stopping rule in Step 2 follows from the fact that $w^* = 0$ for VI problems (see, e.g., Zuhovickii et al. 1969 and Auslender 1976). Theorem 11 and 12 below describe how to obtain the solution to the VI when CPVI terminates finitely and when it generates an infinite sequence, respectively.

Theorem 11: If $w^k = 0$ and $F(x^j)(x^j - u^k) = 0$, then x^j solves the VI problem.

Proof: Under the strong monotonicity assumption, the solution to the VI problem must be unique [see, e.g., Auslender (1976) and Hearn et al. (1984)]. Thus, there must be only one x^j such that

$$0 = w^k = F(x^j) (x^j - u^k)$$

and for $i \neq j$

$$0 = w^k < F(x^i) (x^i - u^k).$$

Then, the kth master problem can be reduced to a problem with only one cut, i.e.,

$$0 = w^k = \max \quad w$$

$$\text{s.t.} \quad w \leq F(x') (x' - u)$$

$$u \in S,$$

or

$$0 = w^k = \max_{u \in S} F(x^j)(x^j \cdot u).$$

However,

$$0 = \max_{u \in S} F(x^j)(x^j \cdot u) \geq F(x^j)(x^J \cdot u) \qquad \forall u \in S$$

Multiply through by –1 to obtain

$$F(x^j)(u \cdot x^J) \geq 0 \qquad \forall u \in S.$$

That is, x^j solves the VI problem. $\square$

Theorem 12: The solution to the VI problem, u^*, is a limit point of the set $[X^\infty]$.

Proof: For each k, let $i(k)$ denote an index of an active cut for the kth master problem, i.e., $i(k)$ satisfies

$$w^k = F(x^{i(k)})x^{i(k)} \cdot F(x^{i(k)})u^k.$$

Then, we have

$$\alpha \left| x^{i(k)} \cdot u^* \right|^2 \leq (F(x^{i(k)}) \cdot F(u^*))(x^{i(k)} \cdot u^*)$$

$$= F(x^{i(k)})(x^{i(k)} \cdot u^*) \cdot F(u^*)(x^{i(k)} \cdot u^*)$$

$$\leq F(x^{i(k)})(x^{i(k)} \cdot u^*)$$

$$= w^k$$

The first inequality follows from the definition of strong monotonicity and the second from the fact that u^* solves the VI problem, i.e.,

31

$$F(u^*)(x - u^*) \geq 0 \qquad\qquad \forall x \in S.$$

Since S is compact, there must exist a subset $K \subseteq \{0, 1, 2, \ldots\}$ satisfying the condition in Lemma 9, thereby having the property that

$$0 = w^* = \lim_{k \in K} w^k$$

Thus, from the above sequence of inequalities, it must be true that

$$\lim_{k \in K} \left| x^{i(k)} - u^* \right|^2 = 0 \quad or \quad \lim_{k \in K} x^{i(k)} = u^*.$$

$\square$

From the above theorem, there exists a subsequence of active cuts, $\{x^{i(k)}\}$, converging to u^*. Alternately, Lemma 10 also implies that the sequence $\{u^k\}$ also converges to u^* when it 'strongly' solves the VI problem (see Figure 4), i.e.,

$$F(v)(x - u^*) > 0 \qquad\qquad \forall x \in S \quad \& \quad x \neq u^*.$$

In this case, the problem: $\max\{L^\infty(u) : u \in S\}$ admits a unique solution. To verify this, assume that $u' \neq u^*$ is an alternate solution to the problem. Then

$$L^\infty(u') = \min_{x \in [X^\infty]} \{F(x)(x - u')\} \leq F(u^*)(u^* - u') < 0.$$

where the first inequality follows from the fact that $u^* \in [X^\infty]$ and the second from the fact that u^* 'strongly' solves the VI problem. However, from Lemma 10, u^* also solves $\max\{L^\infty(u) : u \in S\}$ and $L^\infty(u^*) = w^* = 0$. Thus, the above inequalities is a contradiction since $L^\infty(u^*) = L^\infty(u') = 0$. Therefore, when u^* 'strongly' solves the VI problem, the entire sequence $\{u^k\}$ must converge to u^*, since every convergent subsequence of $\{u^k\}$ must converge to a solution of the problem $\max\{L^\infty(u) : u \in S\}$ which has a unique solution (see, page 234 of Bazaraa and Shetty, 1979).
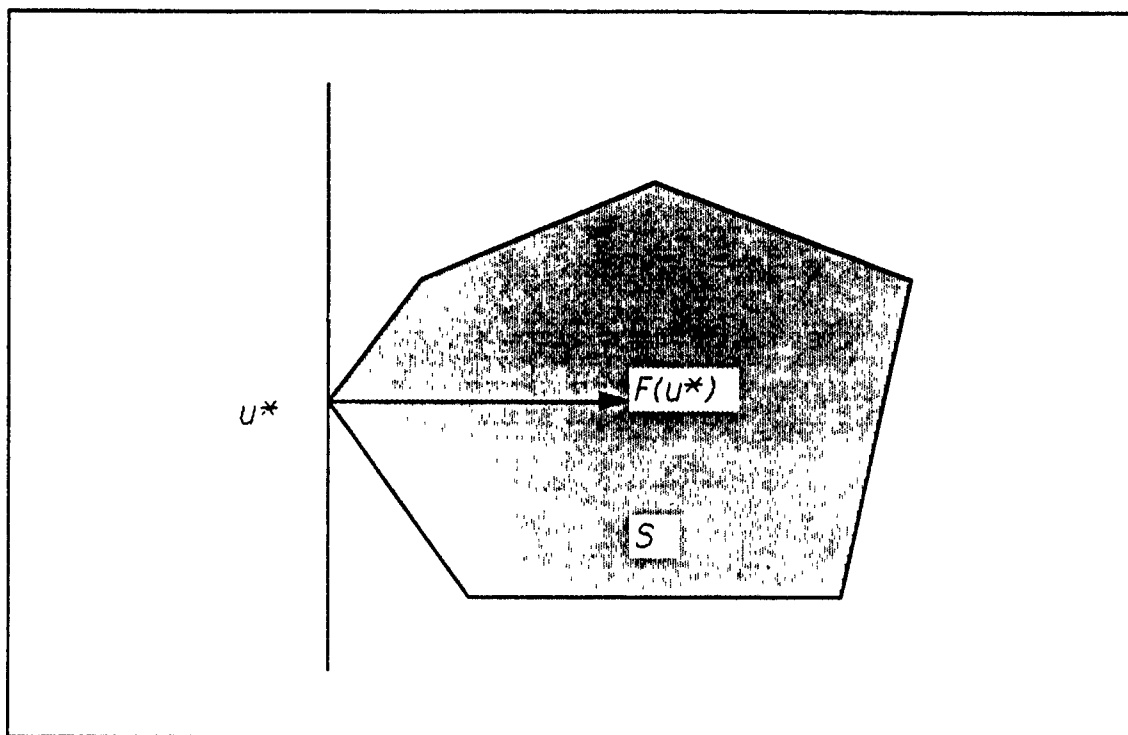
Figure 4: A 'Strong' Solution to a Variational Inequality Problem

The rate of convergence of CPVI depends in part on the choice for x^k in Step 2. The choice requiring the least effort is due to Zuhovickii et al. (1979); they set $x^k = u^k$. Under this choice

$$F(x^k)(x^k - u^k) = F(u^k)(u^k - u^k) = 0.$$

Nguyen and Dupuis (1984) proposed another choice of x^k. They let x^k be the solution to the VI problem over the line segment joining x^{k-1} and u^k, denoted as $\ell[x^{k-1}, u^k]$. Thus, $x^k \in \ell[x^{k-1}, u^k]$ and satisfies

$$F(x^k)(y - x^k) \geq 0 \qquad \forall \, y \in \ell[x^{k-1}, u^k].$$

Since $u^k \in \ell[x^{k-1}, u^k]$,

$$F(x^k)(u^k - x^k) \geq 0 \quad \text{or} \quad F(x^k)(x^k - u^k) \leq 0.$$

So, Nguyen and Dupuis choice of x^k satisfies the requirement in Step 2.
Figure 5 illustrates this choice of x^k. In Figure 5(a), if $F(x^k)$ is a gradient of a
function, say, $f(x)$, then x^k would be the unconstrained minimizer of a one-
dimensional optimization problem. In Figure 5(b), the 'unconstrained
minimizer' is to the left of u^k. So, x^k must be set equal to u^k. However, this
suggests a generalization of the choice for x^k by Nguyen & Dupuis.

Let

$$\alpha_{up} = \max\left\{\alpha : \alpha > 0 \text{ and } x^{k-1} + \alpha(u^k - x^{k-1}) \in S\right\}$$

and set

$$v_{up} = x^{k-1} + \alpha_{up}(u^k - x^{k-1}).$$

Then, choose $x^k \in \ell[v_{up}, x^{k-1}]$ such that

$$F(x^k)(y - x^k) \geq 0 \quad \forall y \in \ell[v_{up}, x^{k-1}].$$

Since $u^k \in \ell[v_{up}, x^{k-1}]$,

$$F(x^k)(u^k - x^k) \geq 0 \quad \text{or} \quad F(x^k)(x^k - u^k) \leq 0.$$

Thus, the new choice of x^k also satisfies the requirement in Step 2. Figure 6
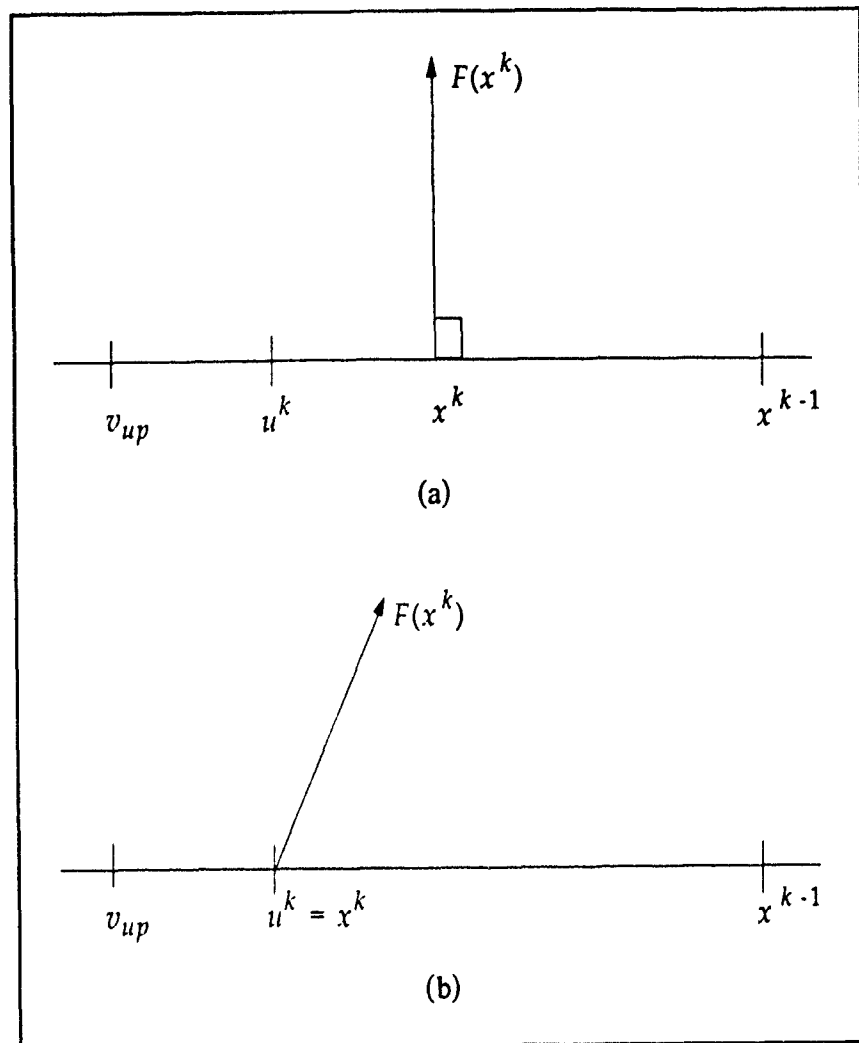illustrates the new choice of x^k.

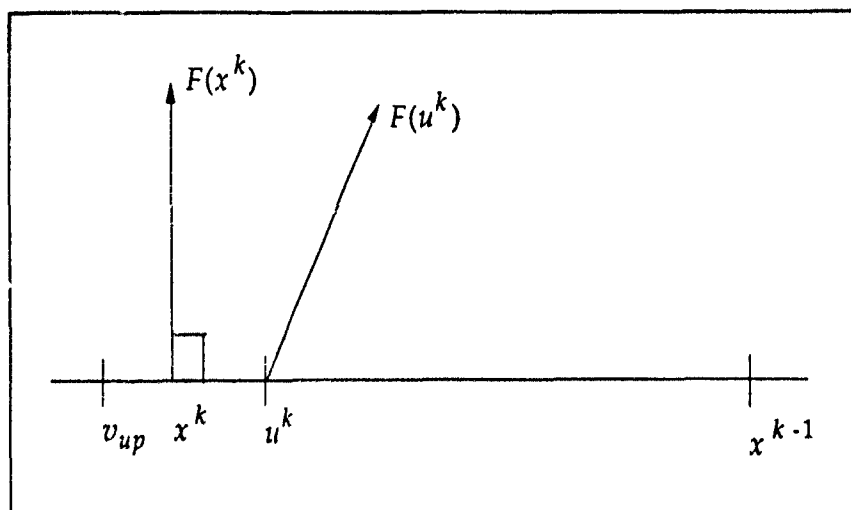Figure 5. Nguyen & Dupuis's Choice for x^k

Figure 6. The New Choice for x^k

5. CONCLUSION

This paper addresses the cutting plane algorithm which has been proposed for problems derived from two different but related areas in operations research: optimization and equilibrium problems. In optimization, the cutting plane algorithm is often used to solve the master problem from Benders decomposition or Lagrangian dual of a nonlinear program. For the equilibrium problem, the algorithm has been applied only to finite dimensional variational inequalities. To unify ideas in the two areas, this paper views problems in both areas as maximin problems. This establishes a common framework for analyzing and examining convergence properties of various schemes for enhancing the cutting plane algorithm. In particular, the analysis also leads to several interesting new results. First is the relationship between adding a line search step and generating 'deep' or pareto optimal cuts. Second are the concept of generating nontangential cuts and the justification for solving the subproblem approximately. The last are an alternate proof of convergence and a generalization of an existing method for generating cuts in the case of variational inequalities.

It has been well demonstrated that the acceleration strategies discussed herein actually reduced the computing time. Hearn and Lawphongpanich (1989a & b) compared the CP algorithm against CPLS on 25 Linear and 25 quadratic integer programming problems. The number of variables in these problem ranges from 20 to 100 with number of constraints ranges from 10 to 50. They concluded that the addition of line search saves cpu time on the average 40% for linear problems and 70% for quadratic. The rather large saving for the quadratic case is due to the fact that the line search only require

two dual function evaluations. In Hearn and Lawphongpanich (1990), the same comparison was made on a traffic assignment problem from Goffin (1987) with 22 arcs, 14 nodes, and 23 origin-destination pairs. The saving due to line search is approximately 50%.

As for variational inequalities, its subproblem is generally a nonlinear program. Thus, by entirely eliminating the subproblem from the steps of the CP algorithm, CPVI can only accelerate the basic algorithm. To demonstrate that CPVI is competitive with existing methods for variational inequalities, Nguyen and Dupuis (1984) compared CPVI with $x^k = v^*$ against the well-regarded Frank-Wolfe algorithm (see, Frank and Wolfe, 1956 and LeBlanc et al., 1975) for the traffic assignment problem. They showed that CPVI with $x^k = v^*$ compares favorably with Frank-Wolfe on problems with 19 arcs, 11 nodes and 4 origin-destination pairs to problems with 2836 arcs, 1052 nodes and 147 origin-destination pairs. Also shown is the fact that CPVI with $x^k = v^*$ is superior to the one with $x^k = u^k$. This is not unexpected since setting $x^k = v^*$ is tantamount to performing a line search along the direction $u^k - x^{k-1}$ (see Hammond, 1984).

REFERENCES

Auslender, A., (1976), *Optimization: Méthod Numériques*, Mason, Paris.

Bazaraa, M. S., Jarvis, J. J., and Sherali, H. D., (1990), *Linear Programming and Network Flows*, Wiley, New York.

Bazarra, M. S., and Shetty, C. M., (1979), *Nonlinear Programming: Theory and Algorithms*, Wiley, New York.

Cheney, E. W., and Goldstein, A. A., (1959), "Newton's Method of Convex Programming and Chebyshev Approximation," *Numerische Mathematik*, 1, 253-268.

Dantzig, G. B., (1963). *Linear Programming and Extensions*, Princeton University Press, Princeton, NJ.

Fisher, M. L. (1981) "The Lagrangian Relaxation Method for Solving Integer Programming Problems," *Management Science* 27, 1-18.

Frank, M. and Wolfe, P., (1956), "An Algorithm for Quadratic Programming," *Naval Research Logistic Quarterly* 3, 95-110.

Fisher, M. L., and Shapiro, J. F., (19/1), "Constructive Quality in Integer Programming," *SIAM Journal on Applied Mathematics*, 27, 31-52.

Hammond, J., (1984), "Solving Asymmetric Variational Inequality Problems and Systems of Equations with Generalized Nonlinear Programming Algorithms," Ph.D. Dissertation, Massachusettes Institute of Technology, Cambridge, Massachusettes.

Harker, P. T., and Pang, J.-S., (199) "Finite-Dimensional Variational Inequality and Nonlinear Complementarity Problems. A Survey of Theory, Algorithms, and Applications," *Mathematical Programming*, 48, No. 2, 161-220.

Hearn, D. W., and Lawphongpanich, S., (1989a) "Lagrangian Dual Ascent by Generalized Linear Programming," *Operations Research Letters*, 8, 189-196.

Hearn, D. W. and Lawphongpanich, S., (1989b), "Generalized Linear Programming with line searches," *Proceedings of the 285h IEEE Conference on Decision and Control*.

Hearn, D. W. and Lawphongpanich, S., (1990), "A Dual Ascent Algorithm for Traffic Assignment Problems," *Transportation Research*, Vol. 24, 423-430.

Hearn, D. W., Lawphongpanich, S., and Nguyen S., (1984), "Convex Programming Formulations of the Asymmetric Traffic Assignment Problem," *Transportation Research*, 18B, 357-365.

Kelly, J. E., (1960), "The Cutting-Plane Method for Solving Convex Programs," *Journal of SIAM*, VIII, No. 4, 703-712.

LeBlanc, L., Morlok, E., and Pierskala, W., (1975), "An Efficient Approach to Solving the Road Network Equilibrium Traffic Assignment Problem," *Transportation Research* 9, 309-318.

Magnanti, T. L., and Wong, R. I., (1981), "Accelerating Benders Decomposition: Algorithmic Enhancement and Model Selection Criteria," *Operations Research*, 29, 464-484.

Magnanti, T. L., Shapiro, J. F. and Wagner, M H. (1976), "Generalized Linear Programming Solves the Dual," *Management Science*, 22, 1195-1203.

Nguyen, S. and Dupuis, C., (1984), "An Efficient Method for Computing Traffic Equilibria in Networks with Asymmetric Transportation Costs," *Transportation Science*, 18, 185-202.

Polyak, B. T., (1969), "Minimization of Unsmooth Functionals," *USSR Comput. Math. and Math. Phys.* 9, 509-521.

Rockafellar, R. T., (1970), *Convex Analysis*, Princeton University Press, Princeton, NJ.

Wets, R. J.-B., (1980), "Convergence of Convex Functions, Variational Inequalities and Convex Optimization Problems,' *Variational Inequalities and Complementarity Problem*, R.W. Cottle, F. Giannessi and J.-L. Lions (Eds.), John Wiley, New York, NY, 375-403.

Wets, R. J.-B., (1983), "Stochastic Programming; Solution Techniques and Approximation,' *Mathematical Programming: The State of the Art*, A. Bachem, M. Grotschel and B. Korte (Eds.), Springer-Verlag, New York, NY, 566-604.

Zangwill, W. I., (1969), *Nonlinear Programming: A Unified Approach*, Prentice-Hall, Englewood Cliffs, NJ.

Zuhovickii, S. I., Poljak, R. A., and Primak, M. E., (1969), "Two Methods of Search for Equilibrium Points of n-Person Concave Games," *Soviet Mathematics Doklady*, 10, No. 2, 279-282.

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